

Exercise 3

Use residues to derive the integration formulas in Exercises 1 through 5.

$$\int_0^{\infty} \frac{\cos ax}{(x^2 + b^2)^2} dx = \frac{\pi}{4b^3}(1 + ab)e^{-ab} \quad (a > 0, b > 0).$$

Solution

The integrand is an even function of x , so the interval of integration can be extended to $(-\infty, \infty)$ as long as the integral is divided by 2.

$$\int_0^{\infty} \frac{\cos ax}{(x^2 + b^2)^2} dx = \int_{-\infty}^{\infty} \frac{\cos ax}{2(x^2 + b^2)^2} dx$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{e^{iaz}}{2(z^2 + b^2)^2},$$

and the contour in Fig. 99. Singularities occur where the denominator is equal to zero.

$$\begin{aligned}(z^2 + b^2)^2 &= 0 \\ z^2 + b^2 &= 0 \\ z &= \pm ib\end{aligned}$$

The singular point of interest to us is the one that lies within the closed contour, $z = ib$.

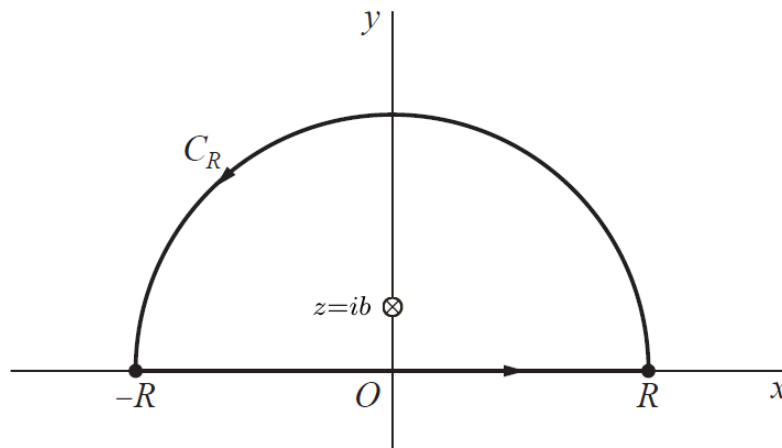


Figure 1: This is Fig. 99 with the singularity at $z = ib$ marked.

According to Cauchy's residue theorem, the integral of $e^{iaz}/[2(z^2 + b^2)^2]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{e^{iaz}}{2(z^2 + b^2)^2} dz = 2\pi i \operatorname{Res}_{z=ib} \frac{e^{iaz}}{2(z^2 + b^2)^2}$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{e^{iaz}}{2(z^2 + b^2)^2} dz + \int_{C_R} \frac{e^{iaz}}{2(z^2 + b^2)^2} dz = 2\pi i \operatorname{Res}_{z=ib} \frac{e^{iaz}}{2(z^2 + b^2)^2}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: \quad z &= r, & r &= -R \rightarrow r = R \\ C_R: \quad z &= Re^{i\theta}, & \theta &= 0 \rightarrow \theta = \pi \end{aligned}$$

As a result,

$$\int_{-R}^R \frac{e^{iar}}{2(r^2 + b^2)^2} dr + \int_{C_R} \frac{e^{iaz}}{2(z^2 + b^2)^2} dz = 2\pi i \operatorname{Res}_{z=ib} \frac{e^{iaz}}{2(z^2 + b^2)^2}.$$

Take the limit now as $R \rightarrow \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{e^{iar}}{2(r^2 + b^2)^2} dr = 2\pi i \operatorname{Res}_{z=ib} \frac{e^{iaz}}{2(z^2 + b^2)^2}.$$

The denominator can be written as $2(z^2 + b^2)^2 = 2(z + ib)^2(z - ib)^2$. From this we see that the multiplicity of the $z - ib$ factor is 2. The residue at $z = ib$ can then be calculated by

$$\operatorname{Res}_{z=ib} \frac{e^{iaz}}{2(z^2 + b^2)^2} = \frac{\phi^{(2-1)}(ib)}{(2-1)!} = \phi'(ib),$$

where $\phi(z)$ is equal to $f(z)$ without the $z - ib$ factors.

$$\phi(z) = \frac{e^{iaz}}{2(z + ib)^2}$$

Take a derivative of it to find $\phi'(z)$.

$$\phi'(z) = \frac{e^{iaz}(iaz - ab - 2)}{2(z + ib)^3} \Rightarrow \phi'(ib) = \frac{e^{ia(ib)}[ia(ib) - ab - 2]}{2(2ib)^3} = \frac{e^{-ab}(-2ab - 2)}{16i^3b^3} = \frac{e^{-ab}(ab + 1)}{8ib^3}$$

So then

$$\operatorname{Res}_{z=ib} \frac{e^{iaz}}{2(z^2 + b^2)^2} = \frac{e^{-ab}(1 + ab)}{8ib^3}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{iar}}{2(r^2 + b^2)^2} dr &= 2\pi i \left[\frac{e^{-ab}(1 + ab)}{8ib^3} \right] \\ \int_{-\infty}^{\infty} \frac{\cos ar + i \sin ar}{2(r^2 + b^2)^2} dr &= \frac{\pi}{4b^3}(1 + ab)e^{-ab} \\ \int_{-\infty}^{\infty} \frac{\cos ar}{2(r^2 + b^2)^2} dr + i \int_{-\infty}^{\infty} \frac{\sin ar}{2(r^2 + b^2)^2} dr &= \frac{\pi}{4b^3}(1 + ab)e^{-ab}. \end{aligned}$$

Match the real and imaginary parts of both sides.

$$\int_{-\infty}^{\infty} \frac{\cos ar}{2(r^2 + b^2)^2} dr = \frac{\pi}{4b^3}(1 + ab)e^{-ab} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin ar}{2(r^2 + b^2)^2} dr = 0$$

Therefore, changing the dummy integration variable to x ,

$$\boxed{\int_0^{\infty} \frac{\cos ax}{(x^2 + b^2)^2} dx = \frac{\pi}{4b^3}(1 + ab)e^{-ab}.}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \rightarrow \infty$. The parameterization of the semicircular arc in Fig. 99 is $z = Re^{i\theta}$, where θ goes from 0 to π .

$$\begin{aligned} \int_{C_R} \frac{e^{iaz}}{2(z^2 + b^2)^2} dz &= \int_0^\pi \frac{e^{iaRe^{i\theta}}}{2[(Re^{i\theta})^2 + b^2]^2} (Rie^{i\theta} d\theta) \\ &= \int_0^\pi \frac{e^{iaR(\cos\theta + i\sin\theta)}}{(R^2e^{i2\theta} + b^2)^2} \left(\frac{Rie^{i\theta}}{2} d\theta \right) \\ &= \int_0^\pi \frac{e^{iaR\cos\theta} e^{-aR\sin\theta}}{(R^2e^{i2\theta} + b^2)^2} \left(\frac{Rie^{i\theta}}{2} d\theta \right) \end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned} \left| \int_{C_R} \frac{e^{iaz}}{2(z^2 + b^2)^2} dz \right| &= \left| \int_0^\pi \frac{e^{iaR\cos\theta} e^{-aR\sin\theta}}{(R^2e^{i2\theta} + b^2)^2} \left(\frac{Rie^{i\theta}}{2} d\theta \right) \right| \\ &\leq \int_0^\pi \left| \frac{e^{iaR\cos\theta} e^{-aR\sin\theta}}{(R^2e^{i2\theta} + b^2)^2} \left(\frac{Rie^{i\theta}}{2} \right) \right| d\theta \\ &= \int_0^\pi \frac{|e^{iaR\cos\theta}| |e^{-aR\sin\theta}|}{|(R^2e^{i2\theta} + b^2)^2|} \left| \frac{Rie^{i\theta}}{2} \right| d\theta \\ &= \int_0^\pi \frac{e^{-aR\sin\theta}}{|R^2e^{i2\theta} + b^2|^2} \frac{R}{2} d\theta \\ &\leq \int_0^\pi \frac{e^{-aR\sin\theta}}{(|R^2e^{i2\theta}| - |b^2|)^2} \frac{R}{2} d\theta \\ &= \int_0^\pi \frac{e^{-aR\sin\theta}}{(R^2 - b^2)^2} \frac{R}{2} d\theta \\ &= \int_0^\pi \frac{e^{-aR\sin\theta}}{\left(1 - \frac{b^2}{R^2}\right)^2} \frac{d\theta}{2R^3} \end{aligned}$$

Now take the limit of both sides as $R \rightarrow \infty$.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{e^{iaz}}{2(z^2 + b^2)^2} dz \right| \leq \lim_{R \rightarrow \infty} \int_0^\pi \frac{e^{-aR\sin\theta}}{\left(1 - \frac{b^2}{R^2}\right)^2} \frac{d\theta}{2R^3}$$

Because the limits of integration do not depend on R , the limit may be brought inside the integral.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{e^{iaz}}{2(z^2 + b^2)^2} dz \right| \leq \int_0^\pi \lim_{R \rightarrow \infty} \frac{e^{-aR\sin\theta}}{\left(1 - \frac{b^2}{R^2}\right)^2} \frac{d\theta}{2R^3}$$

Since θ lies between 0 and π , the sine of θ is positive. a is also positive. Thus, the exponent of e tends to $-\infty$, and the integral tends to zero.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{e^{iaz}}{2(z^2 + b^2)^2} dz \right| \leq 0$$

The magnitude of a number cannot be negative, and only zero has a magnitude of zero. Therefore,

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{e^{iaz}}{2(z^2 + b^2)^2} dz \right| = 0 \quad \rightarrow \quad \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{iaz}}{2(z^2 + b^2)^2} dz = 0.$$